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ASYMPTOTIC BEHAVIOUR FOR WALL POLYNOMIALS AND THE ADDITION FORMULA FOR LITTLE q-LEGENDRE POLYNOMIALS*

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Abstract. Wall polynomials $W_n(x; b, q)$ are considered and their asymptotic behaviour is described when $q = c^{1/n}$ and n tends to infinity. The results are then used to derive the addition and product formulas for the Legendre polynomials from the recently obtained addition and product formulas for little q-Legendre polynomials.

Key words. Wall polynomials, addition formula, product formula, basic hypergeometric polynomials, Legendre polynomials

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1. Introduction. The Wall polynomials $W_n(x; b, q)$ are defined by the recurrence formula

$$W_{n+1}(x; b, q) = \{x - [b + q - (1+q)bq^n]q^n\} W_n(x; b, q) - b(1-q^n)(1-bq^{n-1})q^{2n} W_{n-1}(x; b, q), \qquad n = 0, 1, 2, \cdots \tag{1.1}$$

with initial values $W_{-1} = 0$ and $W_0 = 1$. Clearly $W_n(x; b, q)$ is a monic polynomial of degree n in the variable x. Some properties of Wall polynomials are given in Chihara's book [4, p. 198]. These polynomials are closely related to the continued fraction

$$1 + \frac{x}{1+} \; \frac{(1-b)qx}{1} \; \frac{(1-q)bqx}{+ \quad 1} \; \frac{(1-bq)q^2x}{+ \quad 1} \; \cdots,$$

which was studied by H. S. Wall [16]. The Wall polynomials were also studied by Chihara [5] because they have a Brenke-type generating function, i.e.,

$$\sum_{n=0}^{\infty} W_n(x; b, q) \frac{z^n}{(b; q)_n (q; q)_n} = A(z) B(zx),$$

where

$$A(z) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{z^n}{(q; q)_n} = (zq; q)_\infty,$$

$$B(z) = \sum_{n=0}^{\infty} \frac{z^n}{(b; q)_n (q; q)_n}.$$

We have used the notation

$$(b; q)_n = (1-b)(1-bq) \cdots (1-bq^{n-1}),$$

$$(b; q)_\infty = \lim_{n\to\infty} (b; q)_n;$$

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the latter limit exists whenever $|q|<1$. From this generating function we easily find

$$(1.2)\qquad \begin{aligned} W_n(x; b, q) &= (-1)^n (b; q)_n q^{n(n+1)/2} \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_{n-k}(q; q)_k} q^{k(k-1)/2} \frac{(-q^{-n}x)^k}{(b; q)_k} \\ &= (-1)^n (b; q)_n q^{n(n+1)/2} {}_2\phi_1(q^{-n}, 0; b; q, x), \end{aligned}$$

where the q-hypergeometric (or basic hypergeometric [6]) function is defined by

$${}_{r+1}\phi_r(a_1, \cdots, a_{r+1}; b_1, \cdots, b_r; q, z) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_{r+1}; q)_k}{(b_1; q)_k \cdots (b_r; q)_k} \frac{z^k}{(q; q)_k}.$$

If $0<q<1$ and $0<b<1$ then the Wall polynomials are orthogonal with respect to a positive measure supported on the geometric sequence $\{q^n: n=1, 2, 3, \cdots\}$ and we have

$$\sum_{k=0}^{\infty} W_n(q^{k+1}; b, q) W_m(q^{k+1}; b, q) \frac{b^k}{(q; q)_k} = 0, \qquad n \neq m.$$

The orthonormal polynomials are given by

$$(1.3)\qquad w_n(x; b, q) = \frac{q^{-n(n+1)/2}}{\sqrt{b^n (q; q)_n (b; q)_n}} W_n(x; b, q),$$

and they satisfy

$$(1.4)\qquad (b; q)_\infty \sum_{k=0}^{\infty} w_n(q^{k+1}; b, q) w_m(q^{k+1}; b, q) \frac{b^k}{(q; q)_k} = \delta_{n,m}, \qquad n, m \geqq 0$$

and the three-term recurrence relation (1.1) becomes

$$(1.5)\qquad x w_n(x; b, q) = a_{n+1} w_{n+1}(x; b, q) + b_n w_n(x; b, q) + a_n w_{n-1}(x; b, q)$$

with $w_{-1}=0$, $w_0=1$, and

$$(1.6)\qquad \begin{aligned} a_n &= a_n(b, q) = q^n \sqrt{b(1-q^n)(1-bq^{n-1})}, && n = 1, 2, 3, \cdots, \\ b_n &= b_n(b, q) = q^n[b+q-(1+q)bq^n], && n = 0, 1, 2, \cdots. \end{aligned}$$

Sometimes it is convenient to use the notation

$$(1.7)\qquad (b; q)_\infty \sum_{k=0}^{\infty} f(q^{k+1}) \frac{b^k}{(q; q)_k} = \int_0^1 f(z)\, d\mu(z; b, q), \qquad f \in C[0, 1]$$

so that $\mu(\cdot; b, q)$ is the orthogonality measure for the Wall polynomials $W_n(x; b, q)$.

Recently Koornwinder [8] obtained the addition formula for little q-Legendre polynomials by using the fact that the matrix elements of the irreducible unitary representations of the quantum group $S_\mu U(2)$ (see, e.g., Woronowicz [17], [18]) can be expressed in terms of little q-Jacobi polynomials (Masuda et al. [9], Vaksman and Soibelman [13], Koornwinder [7]). The little q-Jacobi polynomials are defined in terms of q-hypergeometric functions by

$$p_n(x; a, b \mid q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx).$$

If $a=q^\alpha$ and $b=q^\beta$ then these little q-Jacobi polynomials approach the Jacobi polynomials $P_n^{(\alpha,\beta)}(1-2x)/P_n^{(\alpha,\beta)}(1)$ as q tends to 1 [1], [3]. If $a=b=1$ then we have the little q-Legendre polynomials. Notice that for $b=0$ we essentially have the Wall polynomials:

$$
\begin{aligned}
p_n\left(\frac{x}{q};\frac{b}{q},0\,\middle|\,q\right) &= (-1)^n \frac{q^{-n(n+1)/2}}{(b;q)_n} W_n(x;b,q) \\
&= (-1)^n \left\{\frac{b^n(q;q)_n}{(b;q)_n}\right\}^{1/2} w_n(x;b,q).
\end{aligned} \tag{1.8}
$$

The addition formula for little q-Legendre polynomials is

$$
\begin{aligned}
&p_m(q^z;1,1\,|\,q)p_y(q^z;q^x,0\,|\,q) \\
&\quad = p_m(q^{x+y};1,1\,|\,q)p_m(q^y;1,1\,|\,q)p_y(q^z;q^x,0\,|\,q) \\
&\quad + \sum_{k=1}^{m} \frac{(q;q)_{x+y+k}(q;q)_{m+k}q^{k(y-m+k)}}{(q;q)_{x+y}(q;q)_{m-k}(q;q)_k^2} p_{m-k}(q^{x+y};q^k,q^k\,|\,q) \\
&\quad \cdot p_{m-k}(q^y;q^k,q^k\,|\,q)p_{y+k}(q^z;q^x,0\,|\,q) \\
&\quad + \sum_{k=1}^{m} \frac{(q;q)_y(q;q)_{m+k}q^{k(x+y-m+1)}}{(q;q)_{y-k}(q;q)_{m-k}(q;q)_k^2} p_{m-k}(q^{x+y-k};q^k,q^k\,|\,q) \\
&\quad \cdot p_{m-k}(q^{y-k};q^k,q^k\,|\,q)p_{y-k}(q^z;q^x,0\,|\,q)
\end{aligned} \tag{1.9}
$$

with $x, y, z = 0, 1, 2, \cdots$. Rahman [11] has given an analytic proof of this addition formula while Rahman and Verma [12] have given similar formulas for the continuous q-ultraspherical polynomials. The right-hand side of the above formula can be considered as an expansion of the left-hand side in terms of Wall polynomials. For $q\uparrow 1$ we should get the familiar addition formula for Legendre polynomials (see, e.g., [2, pp. 29-38]), but this limit involves some interesting asymptotic formulas for the Wall polynomials $W_n(x; b, c^{1/n})$ with $0<c<1$ and n tending to infinity. This was the main reason for investigating such asymptotic formulas for Wall polynomials.

In § 2 we establish some weak asymptotics for Wall polynomials. In § 3 we show how the addition formula for Legendre polynomials can be obtained from the addition formula for little q-Legendre polynomials by letting $q\to 1$, and in § 4 we obtain the familiar product formulas for Legendre polynomials from the product formulas for little q-Legendre polynomials.

2. Weak asymptotics for Wall polynomials. For little q-Jacobi polynomials $p_n(x; a, b\,|\,q)$ we can put $a=q^\alpha$ and $b=q^\beta$ and let $q\uparrow 1$ to find Jacobi polynomials on $[0,1]$. However, if either a or b is zero, which is exactly what happens for Wall polynomials, then the limit as $q\uparrow 1$ is $(1+(x/(a-1)))^n$. Therefore another approach is needed to handle the behaviour of Wall polynomials as $q\uparrow 1$. It turns out that we can find some relevant results if we consider the polynomials $W_n(x; b, c^{1/n})$ for $n\to\infty$. We will prove a more general result for orthonormal polynomials $\{p_k(x;n)\colon k=0,1,2,\cdots;\ n\in\mathbb{N}\}$, where k is the degree of the polynomial and n an extra (discrete) parameter. The recurrence formula for these polynomials is given by

$$
xp_k(x;n) = a_{k+1,n}p_{k+1}(x;n) + b_{k,n}p_k(x;n) + a_{k,n}p_{k-1}(x;n), \tag{2.1}
$$

where $a_{k,n}>0$, $b_{k,n}\in\mathbb{R}$, $p_0(x;n)=1$, and $p_{-1}(x;n)=0$. Orthogonal polynomials with regularly varying recurrence coefficients [15] are of this type.

THEOREM 1. *Assume that $[r, s]$ is a finite interval that, for all n, contains the support of the orthogonality measure for $\{p_k(x; n)\}$. Assume moreover that*

$$\lim_{n\to\infty} a_{n,n} = A > 0, \qquad \lim_{n\to\infty} b_{n,n} = B \in \mathbb{R} \tag{2.2}$$

and that

$$\lim_{n\to\infty} (a_{k,n}^2 - a_{k-1,n}^2) = 0, \qquad \lim_{n\to\infty} (b_{k,n} - b_{k-1,n}) = 0, \tag{2.3}$$

uniformly in k, then

$$\lim_{n\to\infty} \frac{p_{n+1}(x; n)}{p_n(x; n)} = \rho\left(\frac{x-B}{2A}\right), \tag{2.4}$$

uniformly on compact sets of $\mathbb{C}\backslash[r, s]$, where $\rho(x) = x + \sqrt{x^2-1}$ (the square root here is defined to be the one for which $|\rho(x)| > 1$ for $x \in \mathbb{C}\backslash[-1, 1]$).

Proof. Let K be a compact set in $\mathbb{C}\backslash[r, s]$; then the distance between K and $[r, s]$ is strictly positive. Denote this distance by $\delta > 0$. A decomposition into partial fractions gives

$$\frac{p_{k-1}(x; n)}{p_k(x; n)} = a_{k,n} \sum_{j=1}^{k} \frac{d_{j,k}}{x - x_{j,k}},$$

where $\{x_{j,k}: 1 \leqq j \leqq k\}$ are the zeros of $p_k(x; n)$ and $\{d_{j,k}: 1 \leqq j \leqq k\}$ are positive numbers adding up to 1. Since all the zeros of $p_k(x; n)$ are in $[r, s]$ we have $|x - x_{j,k}| > \delta$ for $x \in K$ and therefore

$$\left|\frac{p_{k-1}(x; n)}{p_k(x, n)}\right| < \frac{a_{k,n}}{\delta} \tag{2.5}$$

holds uniformly for $x \in K$. Consider the Turán determinant

$$D_k(x; n) = p_k^2(x; n) - \frac{a_{k+1,n}}{a_{k,n}} p_{k+1}(x; n) p_{k-1}(x; n).$$

By using the recurrence relation (2.1) we find

$$\begin{aligned} D_k(x; n) = D_{k-1}(x; n) &+ \frac{b_{k,n} - b_{k-1,n}}{a_{k,n}} p_k(x; n) p_{k-1}(x; n) \\ &+ \frac{a_{k,n}^2 - a_{k-1,n}^2}{a_{k,n} a_{k-1,n}} p_{k-2}(x; n) p_k(x; n) \end{aligned} \tag{2.6}$$

(see [14, Thm. 4.10, p. 117]). If we define

$$R_{k,n}(x) = \frac{D_k(x; n)}{p_{k+1}(x; n) p_k(x; n)},$$

then by (2.6)

$$\begin{aligned} |R_{k,n}(x)| \leqq |R_{k-1,n}(x)| \left|\frac{p_{k-1}(x; n)}{p_{k+1}(x; n)}\right| &+ \frac{|b_{k,n} - b_{k-1,n}|}{a_{k,n}} \left|\frac{p_{k-1}(x; n)}{p_{k+1}(x; n)}\right| \\ &+ \frac{|a_{k,n}^2 - a_{k-1,n}^2|}{a_{k,n} a_{k-1,n}} \left|\frac{p_{k-2}(x; n)}{p_{k+1}(x; n)}\right|, \end{aligned}$$

so that by (2.5) we have for $x \in K$

$$|R_{k,n}(x)| \leqq \frac{a_{k,n} a_{k+1,n}}{\delta^2} |R_{k-1,n}(x)| + |b_{k,n} - b_{k-1,n}| \frac{a_{k+1,n}}{\delta^2} + |a_{k,n}^2 - a_{k-1,n}^2| \frac{a_{k+1,n}}{\delta^3}.$$

By the conditions imposed there exists a constant C such that $a_{k,n} < C$ for every n and k (cf. [4, Chap. IV, Example 2.12]). Therefore, by (2.3),

$$|R_{k,n}(x)| \leqq \left(\frac{C}{\delta}\right)^2 |R_{k-1,n}(x)| + A_n, \qquad x \in K,$$

where $A_n \to 0$ as $n \to \infty$. Iteration gives

$$|R_{n,n}(x)| \leqq A_n \frac{(C/\delta)^{2n} - 1}{(C/\delta)^2 - 1} + |R_{0,n}(x)|(C/\delta)^{2n}, \qquad x \in K.$$

If $\delta > C$ then obviously $R_{n,n}(x) \to 0$ as $n \to \infty$ (use $|R_{0,n}| = |p_0(x; n)/p_1(x; n)| < a_{1,n}/\delta$), which by (2.2), (2.3), and (2.5) leads to

$$\lim_{n\to\infty} \left| \frac{p_n(x; n)}{p_{n+1}(x; n)} - \frac{p_{n-1}(x; n)}{p_n(x; n)} \right| = 0, \tag{2.7}$$

uniformly for $x \in K$ (provided $\delta > C$). By (2.5) the sequence of analytic functions $p_n(x; n)/p_{n+1}(x; n)$ is uniformly bounded on compact sets of $\mathbb{C}\backslash[r, s]$ and thus there exists a subsequence converging to some function $L(x)$, uniformly on K. Use the recurrence formula (2.1) and the properties (2.2), (2.3), and (2.7) to find that this limit satisfies

$$x = \frac{A}{L(x)} + B + AL(x),$$

and since $|p_n(x; n)/p_{n+1}(x; n)| < C/\delta < 1$ for $x \in K$ by (2.5) we have

$$\frac{1}{L(x)} = \rho\left(\frac{x-B}{2A}\right).$$

This gives the result for $\delta > C$. This can be extended to hold for $\delta > 0$ by using the Stieltjes-Vitali theorem (cf. [4, p. 121]) and the uniform bound (2.5). □

Remark. The asymptotic behaviour actually holds uniformly on compact sets of $\mathbb{C}\backslash\Omega$, where Ω is the closure of the set of zeros of $p_n(x; n)$ as n runs through the integers. Clearly, Ω is a subset of $[r, s]$ since the zeros of $p_n(x; n)$ are all inside the interval $[r, s]$. The condition that the joint supports of the orthogonality measures should be contained in the finite interval $[r, s]$ can also be relaxed. Only the zeros of $p_k(x; n)(k \leqq n+1,\ n = 0, 1, 2, \cdots)$ must lie in $[r, s]$.

COROLLARY 1. *Suppose* $0 < b < 1$ *and* $0 < c < 1$. *Then*

$$\lim_{n\to\infty} \frac{W_{n+k}(x; b, c^{1/n})}{W_n(x; b, c^{1/n})} = \{b(1-c)(1-bc)c^2\}^{k/2} \rho^k \left(\frac{x - [b+1-2bc]c}{2c\sqrt{b(1-c)(1-bc)}}\right) \tag{2.8}$$

uniformly on compact sets of $\mathbb{C}\backslash[0, 1]$.

Proof. The proof follows immediately from

$$\lim_{n\to\infty} \frac{W_{n+k}(x; b, c^{1/n})}{W_{n+k-1}(x; b, c^{1/n})} - \{b(1-c)(1-bc)c^2\}^{1/2} \rho \left(\frac{x - [b+1-2bc]c}{2c\sqrt{b(1-c)(1-bc)}}\right),$$

which in turn can be proved by using Theorem 1 with recurrence coefficients $a_{k,n} = a_k(b, c^{1/n})$ and $b_{k,n} = b_k(b, c^{1/n})$ given by (1.6). □

COROLLARY 2. *Suppose* $0 < b < 1$ *and* $0 < c < 1$. *Then*

$$\lim_{n\to\infty} \frac{p_{n+k}(z; b, 0 \,|\, c^{1/n})}{p_n(z; b, 0 \,|\, c^{1/n})} = (-1)^k \left\{\frac{b(1-c)}{1-bc}\right\}^{k/2} \rho^k \left(\frac{z - [b+1-2bc]c}{2c\sqrt{b(1-c)(1-bc)}}\right) \tag{2.9}$$

uniformly for z on compact subsets of $\mathbb{C}\backslash[0, 1]$, *where* $p_n(x; a, b \mid q)$ *are the little q-Jacobi polynomials.*

Proof. This follows immediately from (1.8) and Corollary 1. □

It is important in the asymptotic formula (2.4) that the variable x stays away from the zeros of $p_n(x; n)$. On the set Ω, the closure of the zeros of $p_n(x; n)$, the orthogonal polynomials will oscillate. The following theorem gives a result about the weak convergence of measures involving the polynomials $p_k(x; n)$ on $[r, s]$ in terms of their orthogonality measures.

THEOREM 2. *Assume that* $[r, s]$ *is a finite interval that, for all n, contains the support of the orthogonality measure* μ_n *for the orthonormal polynomials* $\{p_k(x; n): k = 0, 1, 2, \cdots\}$. *Assume, moreover, that for all* $k \in \mathbb{Z}$

$$\lim_{n\to\infty} a_{n+k,n} = A, \qquad \lim_{n\to\infty} b_{n+k,n} = B; \tag{2.10}$$

then for every continuous function f on $[r, s]$

$$\lim_{n\to\infty} \int_r^s f(z)p_n(z; n)p_{n+k}(z; n)\,d\mu_n(z) = \frac{1}{\pi}\int_{B-2A}^{B+2A} \frac{f(z)T_k((z-B)/(2A))}{\sqrt{4A^2-(z-B)^2}}\,dz,$$

where $T_n(x)$ *are the Chebyshev polynomials of the first kind.*

Proof. We follow the ideas of Nevai and Dehesa [10, Lemma 3]. Let m be a positive integer and apply the recurrence formula (2.1) repeatedly to get

$$z^m p_n(z; n) = \sum_{\substack{-1\le k_i\le 1\\ i=1,2,\cdots,m}} \alpha_{n,n+k_1}\alpha_{n+k_1,n+k_1+k_2}\cdots\alpha_{n+k_1+\cdots+k_{m-1},n+k_1+\cdots+k_m} p_{n+k_1+\cdots+k_m}(z; n),$$

where

$$\alpha_{j,k} = \begin{cases} a_{j,n} & \text{if } k = j-1,\\ b_{j,n} & \text{if } k = j,\\ a_{j+1,n} & \text{if } k = j+1.\end{cases}$$

Hence

$$\int_r^s z^m p_n(z; n)p_{n+k}(z; n)\,d\mu_n(z) = \sum_{\substack{-1\le k_i\le 1\\ i=1,2,\cdots,m\\ k_1+\cdots+k_m=k}} \alpha_{n,n+k_1}\alpha_{n+k_1,n+k_1+k_2}\cdots\alpha_{n+k_1+\cdots+k_{m-1},n+k}.$$

Because of this equation and by (2.10) it follows that the limit as $n\to\infty$ of $\int_r^s z^m p_n(z; n)p_{n+k}(z; n)\,d\mu_n(z)$ is the same as the limit of

$$\frac{1}{2A^2\pi}\int_{B-2A}^{B+2A} z^m U_n\left(\frac{z-B}{2A}\right)U_{n+k}\left(\frac{z-B}{2A}\right)\sqrt{4A^2-(z-B)^2}\,dz$$

since the Chebyshev polynomials of the second kind $U_n((z-B)/2A)$ are the orthogonal polynomials with constant recurrence coefficients $a_n = A$ and $b_n = B$. Use the identity

$$U_n(x)\,U_{n+k}(x) = \frac{1}{2}\,\frac{T_k(x) - T_{2n+k+2}(x)}{1-x^2}$$

to find

$$\frac{1}{2A^2\pi}\int_{B-2A}^{B+2A} z^m U_n\left(\frac{z-B}{2A}\right)U_{n+k}\left(\frac{z-B}{2A}\right)\sqrt{4A^2-(z-B)^2}\,dz$$

$$= \frac{1}{\pi}\int_{B-2A}^{B+2A} \frac{z^m T_k((z-B)/2A)}{\sqrt{4A^2-(z-B)^2}}\,dz - \frac{1}{\pi}\int_{B-2A}^{B+2A}\frac{z^m T_{2n+k+2}((z-B)/2A)}{\sqrt{4A^2-(z-B)^2}}\,dz.$$

If $2n+k+2>m$ then the second term on the right-hand side vanishes because of orthogonality, and thus we have the result when $f(x)=x^m$. The general result follows from the Hahn-Banach theorem: let the operators $L_{k,n}(k, n=0,1,2,\cdots)$, defined on the Banach space $C[r,s]$ of continuous functions equipped with the supremum norm, be given by

$$L_{k,n}f=\int_r^s f(z)p_n(z;n)p_{n+k}(z;n)\,d\mu_n(z).$$

These are uniformly bounded operators because, by Schwarz's inequality and the orthonormality,

$$\begin{aligned}\left|\int_r^s f(z)p_n(z;n)p_{n+k}(z;n)\,d\mu_n(z)\right|^2 &\leqq \int_r^s |f(z)|p_n^2(z;n)\,d\mu_n(z)\int_r^s |f(z)|p_{n+k}^2(z;n)\,d\mu_n(z)\\ &\leqq \|f\|_\infty^2.\end{aligned}$$

Now use Weierstrass's result that the polynomials form a dense subspace of $C[r,s]$. □

COROLLARY 3. *Suppose* $0<b<1$ *and* $0<c<1$. *Then for every continuous function* f *on* $[0,1]$

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 f(z)w_n(z;b,c^{1/n})w_{n+k}(z;b,c^{1/n})d\mu(z;b,c^{1/n})\\ =\frac{1}{\pi}\int_{B-2A}^{B+2A}\frac{f(z)T_k((z-B)/(2A))}{\sqrt{4A^2-(z-B)^2}}\,dz,\end{aligned}$$

where $A=c\sqrt{b(1-c)(1-bc)}$, $B=(b+1-2bc)c$, *and* $T_n(x)$ *are the Chebyshev polynomials of the first kind.*

Proof. The proof follows because the Wall polynomials $w_n(x;b,c^{1/n})$ satisfy the conditions of Theorem 2, with recurrence coefficients $a_{k,n}=a_k(b,c^{1/n})$ and $b_k(b,c^{1/n})$ given by (1.6). □

3. The addition formula. The little q-Legendre polynomials $p_n(z;1,1\,|\,q)$ and the Wall polynomials $p_n(z;a,0\,|\,q)$ are analytic functions of z and the addition formula (1.9) holds for every $z\in\{q^n: n=0,1,2,\cdots\}$ (which is a set with an accumulation point). Therefore it follows that

$$\begin{aligned}&p_m(z;1,1\,|\,q)p_y(z;q^x,0\,|\,q)\\ &\quad=p_m(q^{x+y};1,1\,|\,q)p_m(q^y;1,1\,|\,q)p_y(z;q^x,0\,|\,q)\\ &\qquad+\sum_{k=1}^m\frac{(q;q)_{x+y+k}(q;q)_{m+k}q^{k(y-m+k)}}{(q;q)_{x+y}(q;q)_{m-k}(q;q)_k^2}p_{m-k}(q^{x+y};q^k,q^k\,|\,q)\\ &\qquad\cdot p_{m-k}(q^y;q^k,q^k\,|\,q)p_{y+k}(z;q^x,0\,|\,q)\\ &\qquad+\sum_{k=1}^m\frac{(q;q)_y(q;q)_{m+k}q^{k(x+y-m+1)}}{(q;q)_{y-k}(q;q)_{m-k}(q;q)_k^2}p_{m-k}(q^{x+y-k};q^k,q^k\,|\,q)\\ &\qquad\cdot p_{m-k}(q^{y-k};q^k,q^k\,|\,q)p_{y-k}(z;q^x,0\,|\,q)\end{aligned}\tag{3.1}$$

holds for every $z\in\mathbb{C}$ and $x,y=0,1,2,\cdots$. It is well known that

$$\lim_{q\uparrow 1}p_n(z;q^\alpha,q^\beta\,|\,q)=R_n^{(\alpha,\beta)}(1-2z),\tag{3.2}$$

where $R_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials with the normalization $R_n^{(\alpha,\beta)}(1)=1$, i.e., $R_n^{(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$. Fix b, c in $(0,1)$ such that $\log b/\log c=\beta/\gamma$ with β, γ positive integers, substitute in (3.1) $q=b^{1/(n\beta)}=c^{1/(n\gamma)}$, $x=n\beta$, $y=n\gamma$, and let $n\to\infty$ through the integers. Then by (2.9), (3.1), and (3.2)

$$\begin{aligned}R_m^{(0,0)}(1-2z)&=R_m^{(0,0)}(1-2bc)R_m^{(0,0)}(1-2b)\\&+\sum_{k=1}^{m}\frac{(m+k)!}{(m-k)!(k!)^2}(1-bc)^kc^kR_{m-k}^{(k,k)}(1-2bc)R_{m-k}^{(k,k)}(1-2c)\\&\cdot(-1)^k\left\{\frac{b(1-c)}{1-bc}\right\}^{k/2}\rho^k\left(\frac{z-[b+1-2bc]c}{2c\sqrt{b(1-c)(1-bc)}}\right)\\&+\sum_{k=1}^{m}\frac{(m+k)!}{(m-k)!(k!)^2}(1-c)^k(bc)^kR_{m-k}^{(k,k)}(1-2bc)R_{m-k}^{(k,k)}(1-2c)\\&\cdot(-1)^k\left\{\frac{1-bc}{b(1-c)}\right\}^{k/2}\rho^{-k}\left(\frac{z-[b+1-2bc]c}{2c\sqrt{b(1-c)(1-bc)}}\right).\end{aligned}$$

Now use the formula $T_k(x)=[\rho^k(x)+\rho^{-k}(x)]/2$; then

$$\begin{aligned}R_m^{(0,0)}(1-2z)&=R_m^{(0,0)}(1-2bc)R_m^{(0,0)}(1-2b)\\&+2\sum_{k=1}^{m}(-1)^k\frac{(m+k)!}{(m-k)!(k!)^2}c^k[b(1-c)(1-bc)]^{k/2}\\&\cdot R_{m-k}^{(k,k)}(1-2bc)R_{m-k}^{(k,k)}(1-2c)T_k\left(\frac{z-[b+1-2bc]c}{2c\sqrt{b(1-c)(1-bc)}}\right).\end{aligned}$$

Finally, choose

$$\begin{aligned}1-2z&=xy-\sqrt{1-x^2}\sqrt{1-y^2}\,t,\\1-2bc&=x,\\1-2c&=y;\end{aligned}$$

then

$$\begin{aligned}R_m^{(0,0)}(xy-\sqrt{1-x^2}\sqrt{1-y^2}\,t)&=R_m^{(0,0)}(x)R_m^{(0,0)}(y)\\&+2\sum_{k=1}^{m}(-1)^k\frac{(m+k)!}{(m-k)!(k!)^2}2^{-2k}\{\sqrt{1-x^2}\sqrt{1-y^2}\}^k\\&\cdot R_{m-k}^{(k,k)}(x)R_{m-k}^{(k,k)}(y)T_k(t),\end{aligned}$$

which is the familiar addition formula for Legendre polynomials. By our method of proof this formula only holds for $t\in\mathbb{C}\backslash\mathbb{R}$ (because we use Corollary 2), but since all the functions considered are analytic in t, the result definitely holds for every $t\in\mathbb{C}$.

4. Product formulas. If we multiply both sides of the addition formula (1.9) by $p_{y+k}(q^z;q^x,0\,|\,q)q^{(x+1)z}/(q;q)_z$ and sum from $z=0$ to ∞, then by the orthogonality (1.4) and by (1.8)

$$\begin{aligned}&\sum_{z=0}^{\infty}p_m(q^z;1,1\,|\,q)p_y(q^z;q^x,0\,|\,q)p_{y+k}(q^z;q^x,0\,|\,q)\frac{q^{(x+1)z}}{(q;q)_z}\\&=\frac{(q;q)_{x+y+k}(q;q)_{m+k}q^{k(y-m+k)}}{(q;q)_{x+y}(q;q)_{m-k}(q;q)_k^2}p_{m-k}(q^{x+y};q^k,q^k\,|\,q)p_{m-k}(q^y;q^k,q^k\,|\,q)\\&\cdot\sum_{z=0}^{\infty}p_{y+k}^2(q^z;q^x,0\,|\,q)\frac{q^{(x+1)z}}{(q;q)_z},\end{aligned}$$

which holds whenever $k\in\{0,1,\cdots,m\}$. In terms of orthonormal Wall polynomials we have by (1.8)

$$\begin{aligned} &p_{m-k}(q^{x+y};q^k,q^k\,|\,q)p_{m-k}(q^y;q^k,q^k\,|\,q)\\ &\quad=(-1)^k\frac{(q;q)_{m-k}(q;q)_k^2}{(q;q)_{m+k}}q^{-k(y+k-m)}\left\{q^{-k(x+1)}\frac{(q;q)_y(q;q)_{x+y}}{(q;q)_{y+k}(q;q)_{x+y+k}}\right\}^{1/2}\\ &\quad\cdot(q^{x+1};q)_\infty\sum_{z=0}^{\infty}p_m(q^z;1,1\,|\,q)w_y(q^{z+1};q^{x+1},q)\\ &\quad\cdot w_{y+k}(q^{z+1};q^{x+1},q)\frac{q^{(x+1)z}}{(q;q)_z},\end{aligned}\tag{4.1}$$

which can be considered as a product formula for the little q-Legendre polynomials and which (for $k=0$) is equivalent with the product formula given by Koornwinder [8]. If we use the notation (1.7) then

$$\begin{aligned}&(q^{x+1};q)_\infty\sum_{z=0}^{\infty}p_m(q^z;1,1\,|\,q)w_y(q^{z+1};q^{x+1},q)w_{y+k}(q^{z+1};q^{x+1},q)\frac{q^{(x+1)z}}{(q;q)_z}\\ &\quad=\int_0^1 p_m\left(\frac{z}{q};1,1\,\Big|\,q\right)w_y(z;q^{x+1},q)w_{y+k}(z;q^{x+1},q)\,d\mu(z;q^{x+1},q).\end{aligned}$$

Fix b, c in $(0,1)$ such that $\log b/\log c=\beta/\gamma$ with β and γ positive integers and let $q=b^{1/(n\beta)}=c^{1/(n\gamma)}$, $1+x=n\beta$, $y=n\gamma$. Then as $n\to\infty$ we have by Corollary 3 and by the uniform convergence in (3.2) (keep in mind that $p_m((z/q);1,1\,|\,q)$ is a polynomial of degree m)

$$\begin{aligned}R_{m-k}^{(k,k)}(1-2bc)R_{m-k}^{(k,k)}(1-2c)&=(-1)^k\frac{(m-k)!(k!)^2}{(m+k)!}c^{-k}\{b(1-c)(1-bc)\}^{-k/2}\\ &\quad\cdot\frac{1}{\pi}\int_{B-2A}^{B+2A}R_m^{(0,0)}(1-2z)\frac{T_k((z-B)/2A)}{\sqrt{4A^2-(z-B)^2}}\,dz,\end{aligned}$$

where $A=c\sqrt{b(1-c)(1-bc)}$ and $B=(b+1-2bc)c$. Setting $bc=x$, $c=y$ gives the familiar product formulas for Legendre polynomials:

$$\begin{aligned}R_{m-k}^{(k,k)}(1-2x)R_{m-k}^{(k,k)}(1-2y)&=(-1)^k\frac{(m-k)!(k!)^2}{(m+k)!}\{xy(1-y)(1-x)\}^{-k/2}\\ &\quad\cdot\frac{1}{\pi}\int_{B-2A}^{B+2A}R_m^{(0,0)}(1-2z)\frac{T_k((z-B)/2A)}{\sqrt{4A^2-(z-B)^2}}\,dz,\end{aligned}$$

with $A=\sqrt{xy(1-x)(1-y)}$ and $B=x+y-2xy$.

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